6 *Day of Recko-NING*

PROBLEM

Graph the circle $x^2 + y^2 = 65$ and determine all of its lattice points. A *lattice point* has integer coordinates.

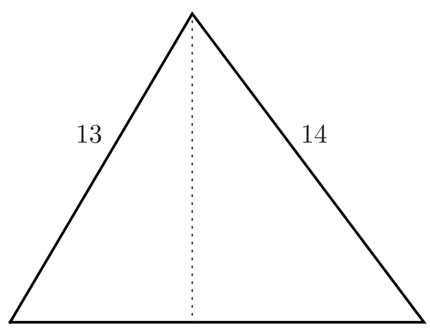
Please don't confuse this with a *lettuce point*, usually found at a salad bar.

Important Stuff.

1. Solve for h and x.

$$\begin{aligned} h^2 + x^2 &= 13^2 \\ h^2 + (15 - x)^2 &= 14^2 \end{aligned}$$

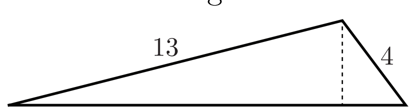
2. Find the perimeter and area of this triangle.



Bill says he will cut you if he sees you using anything with the letter s when solving this problem. The 13-14-15 triangle is called *Superheronian*! It can leap tall buildings in a single bound, but also its side lengths are consecutive integers and its area is an integer.

3. Find the side lengths of a triangle similar to the 13-14-15 triangle whose perimeter and area are equal numerically.

4. Find the area of this triangle.



In no particular order, Jack Bauer, Douglas Adams, Adele, and a calculator appreciate the answers to these triangle problems.

5. The *conjugate* of the complex number $z = a + bi$ is $\overline{z} = a - bi$.

(a) If $z = 5 + 2i$, what is $\overline{z}$?

(b) Let $w = 3 - 4i$. Calculate $w + \overline{w}$ and $w\overline{w}$.

(c) Find a complex number v so that $v + \overline{v} = 14$.

(d) Find a complex number v so that $v\overline{v} = 65$.

(e) Find a complex number v so that $v + \overline{v} = 14$ *and* $v\overline{v} = 65$.

By the way, $w\overline{w}$ is just w multiplied by $\overline{w}$.

6. Find two numbers whose sum is 14 and product is 65.

We're told you can solve this problem by "looking it up".

7. The *magnitude* of the complex number $z = a + bi$ is $|z| = \sqrt{z\overline{z}}$.
(a) Find the magnitude of $z = 5 + 2i$ and of $w = 3 - 4i$.
(b) Rewrite the equation $|z| = \sqrt{(a + bi)(a - bi)}$ as something that doesn't have i in it.
(c) Can the magnitude of a complex number ever be zero? Can it be negative?
(d) Find a complex number whose magnitude is $\sqrt{65}$.

Does anyone else remember the "Day of Reckoning" space from the Game of Life? That game has a spinner because dice are evil. Seriously: the original version came out in 1860 and had a spinning top with the numbers 1 to 6 on it, because dice are evil. The. . . More. . . You. . . Know!

8. How many complex numbers $a + bi$ have integer a, b and magnitude $\sqrt{65}$?

9. (a) Find the magnitude of $5 + 2i$.
(b) Find the magnitude of $(5 + 2i)^2$.
(c) Find a Pythagorean triple with hypotenuse 29.

10. Find a Pythagorean triple with hypotenuse 65.

11. Three positive integers add up to 25 and add up to 360. What are the numbers?

Man, 360 is really divisible! Wouldn't it be awful if there were multiple answers here? Just awful. Seriously.

12. Find the volume and surface area of the box with dimensions $\frac{1}{15}$, $\frac{1}{6}$ and $\frac{1}{4}$.

Neat Stuff.

13. Consider the transformation rule

$$(x, y) \mapsto (5x - 2y, 2x + 5y)$$

For each point, calculate the new point created by this transformation rule.
(a) $P(1, 0)$
(b) $Q(5, 2)$
(c) $R(21, 20)$
(d) $S(65, 142)$

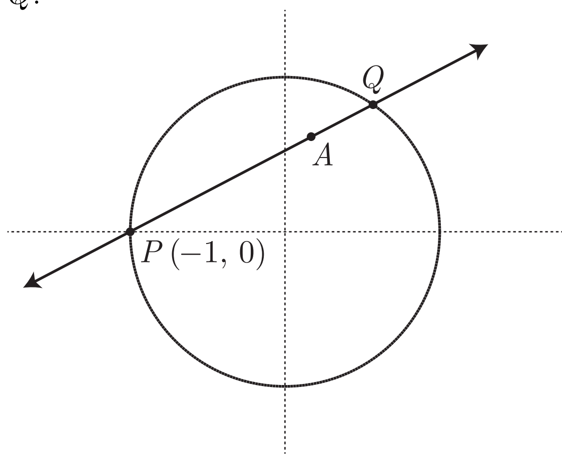
14. The notation $|z|$ that is used for magnitude is the same symbol as the notation used for absolute value. Do real numbers have the same magnitude as their absolute value?

Real numbers *are* complex numbers, they're just numbers like $7 + 0i$ or $-\sqrt{2} + 0i$.

15. A *primitive Pythagorean triple* is a set of three positive integers a, b, c with $a^2 + b^2 = c^2$ where a, b, c share no common factors besides the blatantly obvious.
(a) Find a primitive triple with hypotenuse 65.
(b) Find *both* primitive triples with hypotenuse 85.

6, 8, 10 is *not* one of these, nor is 60, 80, 100.

16. Let P be a point at $(0, -1)$ and A be a point inside the unit circle. Let the line AP intersect the unit circle at point Q.



Calculate $f(A) = AP \cdot AQ$ for each of these points by determining the lengths AP and AQ.

(a) $A = (\frac{1}{2}, 0)$
(b) $A = (-\frac{1}{2}, 0)$
(c) $A = (\frac{1}{3}, -\frac{1}{2})$
(d) $A = (3, 2)$
(e) $A = (-3, 2)$
(f) $A = (x, y)$

17. Find a way to make triangles that have integer side lengths and integer area, and make a bunch of them. *No formulas!*

Bill mad! Bill smash if you use square roots or cosine.

18. Find more Superheronian triangles. Heck, find them all! There's a really small one, but 13-14-15 is the second-smallest.

Useless Stuff.

19. Solve for X:

Honey X don't care!

$$(X^{10} + (\text{mushroom})^2)^4 + X^{10} = \text{snake}$$

20. Solve for Y:

This band inspired today's problem in the box. Or *did* they? No.

$$((\text{I'm}) + Y + (\text{daba}(\text{dee} + \text{di}))^7)^2 = \text{Europop}$$

Tough Stuff.

21. Find all the triangles with integer-length sides whose area and perimeter have the same numerical value. Do they have anything else in common?

22. Find a number n that is the hypotenuse of exactly *four* primitive Pythagorean triples.

23. Same as the last one, but now exactly *eight* primitive Pythagorean triples!

24. Describe what types of positive integers n can be written in the form $n = a^2 + b^2$ for integer a, b. For example, 450 can be written this way: $450 = 21^2 + 3^2$.

25. The number 450 has eight odd factors: 1, 3, 5, 9, 15, 25, 45, 75, 225. Odd factors can be split into factors in the form $4k+1$ and $4k+3$. 450 has 6 odd $4k+1$ factors and 2 odd $4k+3$ factors. It turns out there's a remarkable connection between the number of these types of factors and the *number of different ways* 450 (or any positive integer) can be written in the form $n = a^2 + b^2$ for integer a, b. 450 can be written many different ways ($a = 3, b = 21$ is one of them). Figure out what the rule is—then prove that it works (harder). Don't forget that a or b can be negative or zero.

26. Nobody's found a Matsuura triangle yet. Maybe they don't exist.