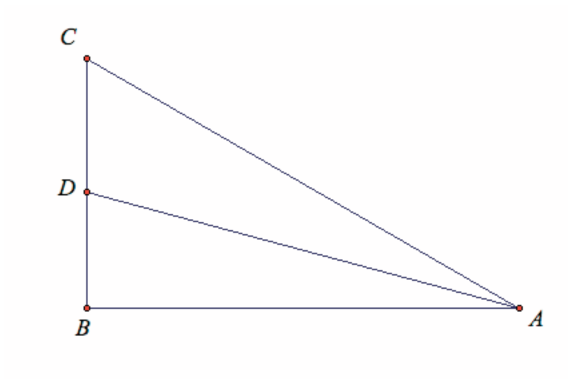
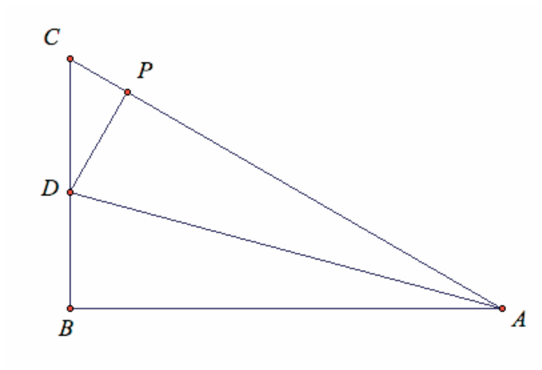
PERSEVERANCE: SALVAGING THE GERM OF AN IDEA

This is a recount of an actual investigation conducted by a group of high school teachers in 2011. There are several morals to the story; one is discussed at the end.

The problem was to find an exact value for $\sin 15°$. After some time, the first solution came from AJ who took a 30-60-90 triangle of hypotenuse length 2 and bisected the 30° angle:



Every trigonometry teacher has seen this: the assumption is that the angle bisector bisects the opposite leg, and from here, you can get a value for $\sin 15°$. Several people pointed out right away that the assumption wasn't correct (and we verified this in dynamic geometry software). But rather than simply calling this a mistake, Jeff got up and said that he saw something. He drew an altitude from D to $\overline{AC}$ and noted that this formed a $\triangle ADP$ that's congruent to $\triangle ADB$ with a little 30-60-90 $\triangle DCP$ leftover.



Jeff said something to the effect that he had no idea if this would help, but it looked promising.

Several people jumped on this idea, arguing along these lines:

- Since $AC = 2$ and $AP = AB = \sqrt{3}$, then $PC = 2 - \sqrt{3}$.
- Since $\triangle DCP$ is 30-60-90, $CD = 2\,PC = 4 - 2\sqrt{3}$.

- But $BC = 1$, so $BD = 1 - CD = 2\sqrt{3} - 3$, and $DP = 2\sqrt{3} - 3$, too.
- The Pythagorean theorem makes $AD = \sqrt{24 - 12\sqrt{3}}$.

Hence

$$\sin 15^\circ = \frac{2\sqrt{3} - 3}{\sqrt{24 - 12\sqrt{3}}}$$

This worked out numerically the same as the calculator approximation, but the feeling was that, to see some structure that might lead to a generalization, this could be simplified considerably. So, groups starting working on this. After a few minutes, Jennifer worked out a board full of algebraic simplifications, stopping at each step to make sure that everyone understood. It was a real *tour de force*, the kind of old-fashioned algebraic calculations that many teachers love. She ended up with

$$\sin 15^\circ = \frac{\sqrt{2 - \sqrt{3}}}{2}$$

This again agreed numerically, and everyone thought it was a great simplification.

Then Kevin entered $\sin 15^\circ$ into a CAS and it produced

$$\frac{\sqrt{6} - \sqrt{2}}{4}$$

Were these two things the same? Pat suggested squaring both, and sure enough, they were.

It was time to go home, but the group felt sure that this idea could be used to get the general half-angle formulas. And, for the next meeting, they decided to investigate how and when one can rewrite $\sqrt{a + \sqrt{b}}$ as the sum of two square roots of rational expressions in a and b.

These results certainly are not new or profound. It is the nature of the work itself—abstracting from numericals, using special cases to inspire generalizations, salvaging false starts—that makes it so faithful to real mathematical work.

The moral: Perseverance has many faces. One of them is that an incorrect conjecture often contains the germ of a good idea. Rather than abandoning a false start, mathematicians often dig into what went wrong, sometimes for long periods of time, seeing if the basic idea can be repaired or used in some other way to come to understanding. It often does. And the repair itself can often be used to launch new investigations—in this case, an investigation into equivalent algebraic expressions. This aspect of perseverance is extremely motivating for learners—the idea that not giving up sometimes leads one from incorrect assumptions to valid results can bootstrap the very practice of "sticking with it."