The Standards for Mathematical Practice in Algebra

1: Make sense of problems and persevere in solving them. There are at least two kinds of sense-making associated with secondary school algebra:

(i) Looking for "hooks" that allow one to hang meaning on algebraic expressions. This often takes the form of connections to geometry, where algebra and geometry talk to each other. A good example of algebra inspiring geometry is when equivalent algebraic expressions have equivalent geometric interpretations. Going the other way, geometry often inspires algebra: Dissection an $a \times b$ rectangle into a square with the same perimeter can lead (with enough numerical prequels) to the algebraic identity

$$\left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2 = ab$$

The formulas $A = \frac{1}{2}(bh)$, $A = (\frac{1}{2}b)h$, and $A = (\frac{1}{2}h)b$ all give the area of a triangle with base b and height h, but the conjure up different dissections of a triangle into a parallelogram.

(ii) At a different end of the sense-making continuum, making sense of algebraic expressions often means treating them as elements of an *algebraic structure*—a system with its own internal logic and in which the expressions have a life of their own. The system of polynomials in two variables with rational coefficients, for example, becomes an object of study in its own right. And identities that are established inside this system are true under any substitution, yielding infinitely many numerical results. Common Core uses the following example: Because the formal identity

$$(a^2 + b^2)^2 = (a^2 - b^2)^2 + (2ab)^2$$

is established via the rules for calculating with polynomials, it is true under any substitution, yielding infinitely many Pythagorean triples.

Perseverance and stamina are often required when working in such systems. For example, it takes intense concentration to see any regularity to the number of factors over the integers for $x^n - 1$ as a function of n.

2: Reason abstractly and quantitatively. Common Core describes one of the hallmarks of algebraic reasoning when it discusses the uses of decontextualization and contextualization. So much of elementary algebra involves creating abstract algebraic models of situations and then transforming the models via algebraic calculations to reveal "hidden" properties of the situations. This connects to the modeling practice, and creating algebraic models of situations often involves abstracting regularity from repeated calculations.

3: Construct viable arguments and critique the reasoning of others. As in geometry, there are central questions in algebra that cannot be answered definitively by checking evidence. There are important results about *all* functions of a certain type—the factor theorem for polynomial functions, for example—and these require general arguments. Another related example:

. . . and understanding that one cannot decide whether or not two functions are equal on an infinite domain by looking at graphs requires a careful critique of what often seems like plausible argument.

deciding whether or not two functions are equal on an infinite set cannot be settled by looking at tables or graphs; it requires arguments of a different sort.

4: Model with Mathematics. Algebra is the classic modeling tool. Because algebraic expressions are statements that can be interpreted in many different algebraic structures, they can be used to describe situations in many different contexts.

A system of linear equations, for example, can tell many different stories.

Modeling with algebra is an area that can tie together many of the other standards (both content and practice). For example, to compare two different text messaging plans, students can define two formulas that produce the cost of each plan as a function of the number of messages sent. Constructing these formulas can be facilitated by calculating the cost for several different input values and then expressing the numerical calculations algebraically, describing the calculation of the cost for *any* numerical input. Finding the break-even point for the two plans amounts to finding the input value for which the two formulas produce the same cost, and this involves solving an equation. Students will be familiar with this solution process, but the solution steps in Algebra 1 will be seen as logical deductions, each justified by a basic rule of arithmetic.

5: Use appropriate tools strategically. One of the most difficult aspects of learning to use algebra fluently is the incorporation of algebraic objects—expressions, functions, systems, and abstract properties of systems in which one can calculate—into the schema of what one considers the "real world." There are two uses of modern technology that are especially useful supports for developing this practice.

- When students build computational models of mathematical functions, they are reviewing, expressing, and getting a chance to examine their own ideas about these functions. At one level, they are getting the benefit that generally comes from writing out one's ideas carefully and in detail: that process, by itself, helps one organize one's thinking, and externalize it enough to review and examine it. But when the "notes" are executable on a calculator, students can run the models they've made, verify their correctness or debug them, and even use them as parts of more complex models. Students who are not yet skilled enough to hold many parts of a model in their heads can build the parts one by one, show how they go together and, for the present, leave the orchestration to the calculator or computer. In short, computers can help students tinker with the physics of mathematics.
- Another use of polynomial algebra is when the "x" is an *indeterminate*. In this view of a polynomial, the letters are just placeholders (rather than "valueholders")—what's really important are the operations *between* the letters. The difference can be illustrated with two common activities in school algebra: simplifying and graphing. When students simplify polynomials, they are thinking of them as formal objects; the fact that $x^2 - 1 = (x-1)(x+1)$ comes from the fact that, if the right side is expanded by "the rules of algebra" you end up with the left side. Computer algebra systems are a perfect tool to help students develop the habit of working in formal systems.

In many parts of algebra, precalculus and calculus, students need to think of functions as *objects* so that they can transform them and calculate with them. This is notoriously difficult for many students—at least as hard as the struggle younger students have when they need to think of fractions as numbers. This model-building is extremely effective as a device that helps students think of functions as things in their own right.

If two polynomials are equivalent formally, they define the same function. In Algebra 2, students see an important converse to this: if two polynomial functions agree at "enough" inputs, one can be obtained from the other by the "rules of algebra."

6: Attend to precision. In algebra, the habit of using precise language is not only a mechanism for effective communication; it's a tool for understanding. Being able to "shoehorn" an idea into a precise algebraic description not only allows you to exploit the idea—it helps you understand the idea in new ways. For example, when investigating loan payments, if students can articulate something like "What you owe at the end of a month is what you owed at the start of the month, plus $\frac{1}{12}^{\text{th}}$ of the yearly interest on that amount, minus the monthly payment," they are well along a path that will let them construct a recursively defined function for calculating loan payments.

7: Look for and make use of structure. In a real sense, algebra is *about* structure. In more advanced courses, algebra is about the structure of systems in which one can calculate. In Algebra 1, students learn to use the structure of algebraic *expressions*. For example, writing $49x^2 + 35x + 6$ as $(7x)^2 + 5(7x) + 6$ highlights the structural similarity between this expression and $z^2 + 5z + 6$, leading to a factorization of the original: $((7x) + 3)((7x) + 2)$.

Seeing hidden structure in expressions is a knack that takes time to develop. For example, it's not easy to see that $x^4 + x^2 + 1$ is a difference of squares "in disguise." Being explicit about "sniffing for patterns" can be a constant and productive theme in algebra class.

This theme continues in Algebra 2, where students delve deeper into transforming expressions in ways that reveal meaning. The example given in Common Core—that $x^4 - y^4$ can be seen as the difference of squares—is typical of this practice. This habit of seeing subexpressions as single entities will serve students well in areas like trigonometry, where, for example, the factorization of $x^4 - y^4$ described above can be used to show that the functions $x \mapsto \cos^4 x - \sin^4 x$ and $x \mapsto \cos^2 x - \sin^2 x$ are, in fact, equal.

There's another kind of structure-seeking that's begun in Algebra 1. Common Core calls for attention to the structural similarities between polynomials and integers. The study of these these similarities can be deepened in Algebra 2: Like integers, polynomials have a division algorithm, and division of polynomials can be used to understand the factor theorem, to transform rational expressions, to help solve equations, and to factor polynomials.

8: Look for and express regularity in repeated reasoning. It's common knowledge among teachers that coming up with equations or functions that model situations is much harder for most students than calculating or transforming the resulting expressions. This mathematical practice is an effective way to help beginning students develop the skill of describing general relationships by working through several specific examples to get the "rhythm" of their calculations and then expressing what they are doing with algebraic symbolism.

Finding the areas of several triangles, given numerical values for their side-lengths, can (if this habit is exercised carefully) lead naturally to a proof of Heron's formula.

This habit can help students make a more complete analysis of sequences, especially arithmetic and geometric sequences, and their associated series. Developing recursive formulas for sequences is facilitated by the practice of abstracting regularity for how you get from one term to the next and then giving a precise description of this process in algebraic symbols. Technology can be a useful tool here: most CAS systems allow one to model recursive function definitions in notation that is close to standard mathematical notation. And spreadsheets make natural the process of taking successive differences ands running totals.

The same thinking—finding and articulating the rhythm in calculations—can help students analyze mortgage payments, and the ability for getting a closed form for a geometric series lets them make a complete analysis of this topic. And this practice is also a tool for using difference tables to find simple functions that agree with a set of data.